Similar Solutions for the Laminar Mixing of Reactive Gases

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Using the two-dimensional compressible laminar boundary-layer equations for a multicomponent gas mixture, the present analysis treats the mixing of two parallel reactive streams under the influence of an externally induced streamwise pressure gradient. The thermodynamic behavior of the gas mixture within the mixing region is specified by an empirical law that matches the density to its exact values at different points across the shear layer. In the analysis, it is assumed that the product of the density and viscosity is constant across the mixing layer, being only a function of its streamwise location. In addition, the usual assumptions of unit Prandtl number and Lewis numbers with either frozen or equilibrium chemistry specifying the state and composition of the gas mixture within the mixing layer are employed. Within the framework of these assumptions, similar solutions of the governing system of equations are obtained for various pressure gradients and velocity ratios of the two reactive streams.

Nomenclature

A_i = constituents of reaction
c_{pi} = specific heat of species i at constant pressure
c_p = specific heat of mixture at constant pressure
f' = velocity ratio parameter, u/u_e
g = stagnation enthalpy ratio parameter, H/H_e
h_i = static enthalpy of ith species
h_i^0 = heat of formation at absolute zero
h = static enthalpy of mixture
H = stagnation enthalpy
k = reaction rate constant
K_p = equilibrium constant expressed in terms of pressure
L_e = Lewis number
p_i = partial pressure
p = static pressure of mixture
P_r = Prandtl number
R = density ratio parameter, ρ_e/ρ
$\mathcal{R}$ = universal gas constant
T = static temperature
u = velocity component in x direction
v = velocity component in y direction
$\dot{w}_i$ = mass rate of production or disappearance of species i
W_i = molecular weight of ith species
W = molecular weight of mixture
x = Cartesian coordinate in flow direction
y = Cartesian coordinate normal to flow direction
z_i = mass fraction ratio parameter, $\tilde{Y}_i/\tilde{Y}_{ie}$
α_{ik} = number of atoms of k in species i
β = pressure gradient parameter
η = transformed y coordinate
κ = quantity defined by Eq. (23)
λ = quantity defined by Eq. (24)
μ = coefficient of viscosity
ν', ν'' = stoichiometric coefficients
ξ = transformed x coordinate
ρ_i = density of ith species
ρ = density of mixture
Y_i = mass fraction of ith species
$\tilde{Y}_k$ = element mass fraction
ϕ = stagnation enthalpy parameter defined in Eq. (24)
ψ = stream function
ν = defined as $(\nu'' - \nu')$

Subscripts

e = values in upper gas stream
f = flame sheet values
i = the ith species
j = values in lower gas stream
k = elements comprising the species
l = a particular reaction
0 = values at $y = 0$

Introduction

In order to properly assess the performance capabilities of advanced air-breathing vehicles utilizing supersonic combustion, it is necessary to accurately describe the aerothermochemical aspects of the gas flow within the combustor. The usual approach to the supersonic combustion problem has been to consider the detonation of premixed fuel and oxidizer as the chemical energy releasing mechanism. It recently has been proposed by Ferri,[1] however, that the use of diffusion or deflagration flames in supersonic combustion exhibits several advantages over the detonation mode of burning usually considered. In this type of burning scheme, the diffusion flames are controlled by the mixing of the injected fuel with the oxidizer, while additional thrust is derived from the injected fuel, without encountering the usual total head losses inherent in the detonation mode of burning.

The analysis of the diffusion or deflagration mode of burning is complicated by the simultaneous occurrence of fuel-oxidizer mixing and chemical reactions that produce large heat release, causing large changes in density and pressure. An analytical treatment of the complete process would require a very complex and cumbersome numerical solution of the viscous flow equations, which may themselves prove to be inadequate to describe properly the processes that occur. As a result, it is necessary to introduce certain simplifying assumptions so that an approximate solution to the problem may be accomplished in an attempt to gain some insight into the mechanics of the complete problem. The usual approaches toward simplifying the problem are either 1) to assume simple chemistry, emphasizing an accurate description of the mixing process or 2) to assume constant pressure mixing, emphasizing a more sophisticated chemical model.

Analytical treatment of laminar mixing problems involving finite reaction rates have been presented by Marble and Adamson,[2] Dooley,[3] and Bloom and Steiger.[4] Recently, Libby and Schetz[5] have treated the laminar wall-slot injection problem for the limiting cases of frozen and equilibrium flow by utilizing the modified Oseen method of Lewis and Carrier.[6] Similar studies of the turbulent mixing of reactive gases have been presented by Libby[7] for the axisymmetric constant pressure mixing of the hydrogen-air combustion system and by Vasiliu[8] for the two-dimensional mixing of a rocket exhausting into an ambient stream. In Ref. 9, simi-

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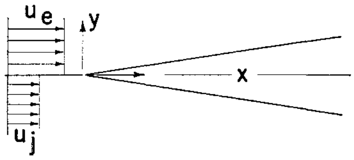

Fig. 1 Schematic representation of flow model.

lar solutions were obtained for the turbulent mixing of two reactive gases under the influence of a streamwise pressure gradient. The analysis presented is valid for either frozen or equilibrium chemistry.

In the present analysis, the laminar-mixing of two reactive gases subjected to a streamwise pressure gradient is considered. The standard boundary-layer equations are utilized to express the viscous behavior of the gas mixture within the mixing region with transport properties selected to correspond to unit values of Prandtl number and Lewis numbers. In addition, the usual assumption is made regarding the behavior of the product of density and viscosity across the viscous layer, namely, that the product of these two gas properties is constant across the layer. By utilizing an empirical relation for the thermodynamic behavior of the gas mixture within the mixing layer, similar solutions of the governing equations are obtained for either frozen or equilibrium flow within the mixing region.

Consideration of a streamwise pressure gradient imposes a restriction on the solution because of the boundary-layer approximations inherent in the basic equations used in the analysis. The severity of this condition is enhanced when similar solutions are desired, since the assumption of similarity imposes certain requirements on the nature of the external gas stream. A study of this restriction for the perfect gas case has been made by Napolitano.[10]

A schematic representation of the flow model considered in this analysis is shown in Fig. 1.

Governing Equations

The basic equations used to describe the steady compressible laminar motion of the gas mixture within the mixing region are the usual laminar boundary-layer equations derived for a multicomponent reacting gas. The validity of these equations to describe properly the process occurring in the mixing layer depends on the assumption that all reaction-induced gradients in the lateral direction be much larger than those in the streamwise direction. Since the gradients normal to a laminar flame front or reaction zone are quite large, it must be required that the angle between the flame front and the axis of mixing be small if the boundary-layer approximations are to be valid throughout the mixing layer.

The most general form of the laminar boundary-layer equations for a reacting gas mixture involving s species are:

Global Conservation of Mass

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{1}$$

Conservation of Momentum

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{2}$$

Conservation of Energy

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\mu}{Pr}\frac{\partial H}{\partial y} + \mu\left(1 - \frac{1}{Pr}\right)\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) + \frac{\mu}{Pr}(L_e - 1)\sum_{i=1}^{s} h_i \frac{\partial Y_i}{\partial y}\right] \tag{3}$$

Conservation of Species

$$\rho u \frac{\partial Y_i}{\partial x} + \rho v \frac{\partial Y_i}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{P_r} L_e \frac{\partial Y_i}{\partial y}\right) + \dot{w}_i \qquad (i = 1, 2, \ldots, s) \tag{4}$$

Each species of the gas mixture is assumed to behave in accordance with the perfect gas equation of state, i.e.,

$$p_i = \rho_i(\mathcal{R}/W_i)T \tag{5}$$

where

$$\sum_{i=1}^{s} p_i = p = \rho\mathcal{R}\sum_{i=1}^{s}\frac{Y_i}{W_i}T = \rho\frac{\mathcal{R}}{W}T \tag{6}$$

The static enthalpy of the gas mixture is given by

$$h = \sum_{i=1}^{s} Y_i h_i = \sum_{i=1}^{s} Y_i\left[\int_0^T c_{p_i} dT + h_i^0\right] \tag{7}$$

whereas the total enthalpy is expressed as

$$H = h + (u^2/2) \tag{8}$$

and the frozen specific heat of the mixture as

$$c_p = \sum_{i=1}^{s} Y_i c_{p_i} = \sum_{i=1}^{s} Y_i \frac{dh_i}{dT} \tag{9}$$

The conservation of species equations can be written in an alternate form by considering the element mass fractions together with the species mass fractions.[11] For any chemical reaction or system of chemical reactions, the number of moles of each element comprising the constituents of the mixture must be conserved. As a result, the element mass fractions for a system of s species and e elements are given by

$$\tilde{Y}_k = \sum_{i=1}^{s} \alpha_{ik}\frac{W_k}{W_i}Y_i \qquad (k = 1, 2, \ldots, e) \tag{10}$$

In addition, since the time rate of change of the element mass fractions is zero, the mass rates of each species in the system are related in accordance with

$$\sum_{i=1}^{s} \alpha_{ik}\frac{\dot{w}_i}{W_i} = 0 \qquad (k = 1, 2, \ldots, e) \tag{11}$$

Equations (10) and (11) are the results of classical chemical kinetics where the chemical system is considered to be in a stationary-isothermal condition with the state and composition of the gas mixture taken as homogeneous throughout the system. The applicability of these relationships to a flowing nonhomogeneous system is justified provided the state and composition of the mixture do not vary appreciably within a characteristic length associated with the chemical reaction times. The intensive variables (pressure, temperature, etc.) describing the thermochemical state of the flowing system are functions of time and position. Consequently, it is necessary to replace the simple time derivatives of classical kinetics by substantial derivatives following a fluid element. Consistent with this assumed chemical flow model, it is possible to replace the conservation of species equations of (4) by

$$\rho u \frac{\partial \tilde{Y}_k}{\partial x} + \rho v \frac{\partial \tilde{Y}_k}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{P_r} L_e \frac{\partial \tilde{Y}_k}{\partial y}\right) \qquad (k = 1, 2, \ldots, e) \tag{12}$$

$$\rho u \frac{\partial Y_i}{\partial x} + \rho v \frac{\partial Y_i}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{P_r} L_e \frac{\partial Y_i}{\partial y}\right) + \dot{w}_i \qquad (i = 1, 2, \ldots s-e) \tag{13}$$

When the volume recombination rates are considered very slow compared to the diffusion rates across streamlines (frozen flow), $\dot{w}_i$ is identically zero, and only Eqs. (13) need be considered for each of the species of the chemical system. When the reaction rates are regarded as very fast in comparison to the convection rates along streamlines or the rates of diffusion across streamlines (equilibrium flow), the existence of finite, nonzero values of $\dot{w}_i$ complicates the conservation of species equations (12) and (13). It is possible in this case, however, to replace Eq. (13) by the equations governing the equilibrium constants of the chemical system. Since the maximum number of independent equilibrium reactions is $(s - e)$, there exist $(s - e)$ equilibrium constants of the form†

$$K_{pl} = K_{pl}(T) = (pW)^{\sum_{i=1}^{s} \nu_{il}} \prod_{i=1}^{s} \left(\frac{Y_i}{W_i}\right)^{\nu_{il}} \qquad (l = 1, 2, \ldots, s-e) \quad (14)$$

for each of the $(s-e)$ independent reactions represented by

$$\sum_{i=1}^{s} \nu_{il}' A_i \xrightarrow{k_l} \sum_{i=1}^{s} \nu_{il}'' A_i \qquad (l = 1, 2, \ldots, s\text{-}e) \quad (15)$$

Consequently, the equilibrium properties of a flowing system can be determined by considering Eqs. (12) and (14) in place of Eq. (4).

Analysis

The basic system of partial differential equations governing the laminar motion of the gas mixture within the mixing region can be transformed to ordinary differential equations provided a natural coordinate system (ξ, η) can be found in which the derivatives of the dependent variables become separable. The solutions desired are such that

$$\chi(x, \eta) = \Lambda(x) f(\eta) \quad (16a)$$

$$u(x, \eta) = u_e(x) f'(\eta) \quad (16b)$$

$$H(x, \eta) = H_e g(\eta) \quad (16c)$$

$$\bar{Y}_i(x, \eta) = \bar{Y}_{ie}(x) z_i(\eta) \quad (16d)$$

where

$$\bar{Y}_i = Y_i \qquad (i = 1, 2, \ldots, s) \quad (17a)$$

for the frozen flow case, and

$$\bar{Y}_i = \tilde{Y}_k \qquad (k = 1, 2, \ldots, e) \quad (17b)$$

for the equilibrium flow case. The transformation required is determined from the necessary conditions that must be imposed in order to reduce the governing system of equations to a one-parameter system. Consistent with this approach, it can be shown that the equations reduce to the desired form by utilizing a change of variables given by

$$\xi = \frac{\Lambda^2(x)}{2} = \int_0^x \rho^* \mu^* u_e dx \quad (18a)$$

$$\eta = \frac{\rho_e u_e}{(2\xi)^{1/2}} \int_0^y \frac{\rho}{\rho_e} dy \quad (18b)$$

and by assuming unit values of Prandtl number and Lewis numbers, constant element and species mass fractions external to the mixing region, and

$$C = \rho\mu/\rho^*\mu^* = 1 \quad (19)$$

where the starred quantities are evaluated at an appropriate reference state. As a result, the governing system of equations becomes

$$f''' + ff'' = (2\xi/u_e)(du_e/d\xi)(f'^2 - R) \quad (20)$$

$$g'' + fg' = 0 \quad (21)$$

$$z_i'' + fz_i' = 0 \quad (22)$$

where the primes denote differentiation with respect to the similarity variable η and $R = \rho_e/\rho$. It is evident from Eqs. (21) and (22) that a Crocco-type integral relation exists between g and z_i provided their boundary conditions are compatible.

The solution of these equations depends on the specification of the coupling existing between the density ratio R, the velocity ratio f', and the stagnation enthalpy ratio g. In general, no simple relationship is available which specifies this mutual dependence other than for a single-component perfect gas. As a result, it becomes necessary to introduce empirical relations that approximately specify the actual coupling of these flow quantities.[12, 13] In the spirit of Hayes[13] and of Hayes and Probstein,[14] it will be assumed that, in general,

$$R(x, \eta) = f'^2(\eta) - \varkappa(x)[f'^2(\eta) - \phi(\eta)] \quad (23)$$

where

$$g(\eta) = \lambda + (1 - \lambda)\phi(\eta) \quad (24)$$

When similar profiles are sought, however, the density ratio R must only be a function of the similarity variable, thereby requiring a constant value for $\varkappa$. The constants λ and $\varkappa$ are determined by matching the density to its actual value at different points across the mixing layer. Since Eq. (23) is automatically satisfied at the outer edge of the layer ($\phi_e = 1$), it must be fitted at two additional points in the mixing region in order that both λ and $\varkappa$ can be evaluated. If the matching points are taken to be the inner boundary ($y \to -\infty$) and the axis of the mixing region ($y = 0$), there results‡

$$\lambda = \frac{(R_j - f_j'^2)(g_0 - f_0'^2) - (R_0 - f_0'^2)(g_j - f_j'^2)}{(R_j - f_j'^2)(1 - f_0'^2) - (R_0 - f_0'^2)(1 - f_j'^2)} \quad (25)$$

$$\varkappa = \frac{(R_0 - f_0'^2)(1 - g_j) - (R_j - f_j'^2)(1 - g_0)}{(1 - f_0'^2)(1 - g_j) - (1 - f_j'^2)(1 - g_0)} \quad (26)$$

In addition to the usual boundary conditions associated with the problem at hand, i.e.,

$$\begin{array}{ll} f' \to 1 & f'' \to 0 \\ \phi \to 1 & \phi' \to 0 \\ z_i \to 1 & z_i' \to 0 \end{array} \quad (27)$$

as $\eta \to +\infty\,(y \to +\infty)$ and

$$\begin{array}{ll} f' \to f_j' & f'' \to 0 \\ \phi \to \phi_j & \phi' \to 0 \\ z_i \to z_{ij} & z_i' \to 0 \end{array} \quad (28)$$

as $\eta \to -\infty\,(y \to -\infty)$, it is required that the pressure gradient be the same above and below the mixing layer. This is a natural condition that arises because of the boundary-layer approximations inherent in the equations used to describe the fluid motion within the mixing region. As a result,

$$\rho_e u_e (du_e/dx) = \rho_j u_j (du_j/dx) \quad (29)$$

or

$$R_j = f_j'^2 = \phi_j \quad (30)$$

Consideration of this condition, together with Eqs. (23) and (24), reduces Eqs. (25) and (26) to

$$\lambda = \frac{g_j - f_j'^2}{1 - f_j'^2} \quad (31)$$

$$\varkappa = \frac{R_0 - f_0'^2}{\phi_0 - f_0'^2} \quad (32)$$

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† Tabulated values of $K_{p_l}(T)$ are available for most of the equilibrium reactions of interest to the problem at hand.

‡ It should be noted that, for a single-component perfect gas, Eqs. (25) and (26) become $\lambda = 0$ and $\kappa = H_e/h_e$, respectively. As a result, Eq. (23) reduces to the well-known perfect gas relationship of $(\rho_e/\rho) - f'^2 = (H_e/h_e)(f'^2 - g)$.

and the governing system of equations for similarity to

$$f''' + ff'' = \beta(f'^2 - \phi) \quad (33)$$

$$\phi'' + f\phi' = 0 \quad (34)$$

with

$$z_i = \frac{1}{(1 - f_j'^2)} [(z_{ij} - f_j'^2) + (1 - z_{ij})\phi] \quad (35)$$

specifying the solutions of the species equations.

Inverse Transformation to Physical Plane

The density ratio R is evaluated from the equation of state once the temperature ratio is determined from the enthalpy-velocity relation

$$\frac{h}{h_e} = \sum_i Y_i \frac{h_i}{h_e} = \frac{H_e}{h_e} [\lambda + (1 - \lambda)\phi - f'^2] + f'^2 \quad (36)$$

since $h_i = h_i(T)$. A convenient analytic representation for $h_i = h_i(T)$ is introduced in the form

$$h_i = C_i T^n \quad (37)$$

where C_i and n are constants selected to best represent the exact $h_i = h_i(T)$ function in the temperature range of interest to the problem at hand. Consequently, it follows that

$$h_i/h_{ie} = (T/T_e)^n \quad (38)$$

which, when utilized in Eq. (36), yields

$$\frac{T}{T_e} = \left[\frac{\lambda + (1 - \lambda)\phi - \tilde{u}_e^2 f'^2}{\Sigma Y_{ie}(Y_i/Y_{ie})(h_{ie}/H_e)}\right]^{1/n} \quad (39)$$

with

$$\tilde{u}_e^2 = u_e^2/2H_e$$

It is evident from Eq. (39) that additional requirements are necessary for similarity of solutions, § i.e.,

$$\tilde{u}_e^2 = \text{const} \rightarrow 1 \qquad h_{ie}/H_e = \text{const}$$

Accordingly, from the equation of state, there results for the density ratio

$$R = \left[\frac{\lambda + (1 - \lambda)\phi - \tilde{u}_e^2 f'^2}{\Sigma Y_{ie}(Y_i/Y_{ie})(h_{ie}/H_e)}\right]^{1/n} \times \left[\frac{\Sigma(Y_i/Y_{ie})(Y_{ie}/W_i)}{\Sigma(Y_{ie}/W_i)}\right] \quad (40)$$

since the molecular weight is given by

$$\frac{1}{W} = \sum_i \frac{Y_i}{W_i} \quad (41)$$

Equation (40), when evaluated at $\eta = 0$, yields the density ratio R_0 required to determine $\varkappa$ from Eq. (32).

Since satisfaction of the pressure gradient condition requires that $R_j = f_j'^2$, the stagnation enthalpy ratio at the inner boundary ($\eta \rightarrow -\infty$) cannot be arbitrarily chosen but must satisfy the relation

$$g_j = \left(f_j'^2 \frac{W_j}{W_e}\right)^n \sum_i Y_{ie} z_{ij} \frac{h_{ie}}{H_e} + \tilde{u}_e^2 f'^2$$

As a result, Eq. (23) together with Eqs. (31) and (32), is equivalent to the equation of state as expressed by Eq. (40). Since Eq. (23) satisfies the equation of state and matches exactly the proper density behavior at the outer boundary, the axis of the mixing region and at the inner boundary, it is felt that it should yield a fairly good representation of the actual density distribution across the mixing layer.

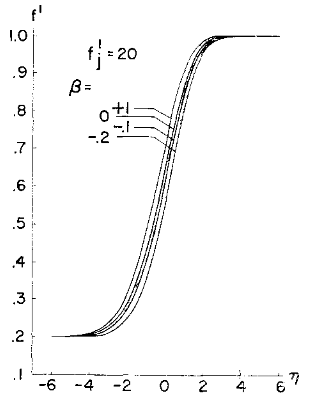

Fig. 2a Velocity profiles vs similarity parameter η ($f_j' = 0.20$).

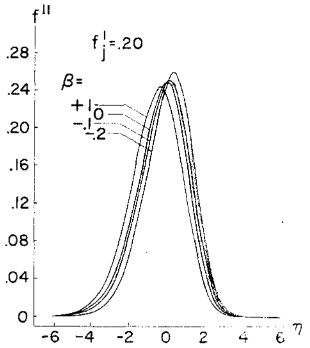

Fig. 2b Shear function vs similarity parameter η ($f_j' = 0.20$).

It should be indicated that, for the zero pressure gradient case, only the solution $f'(\eta)$ need be considered from this analysis. The functions $g(\eta)$ and $z_i(\eta)$ for that case can be obtained as Crocco-type integrals of $f'(\eta)$.

For the frozen flow case, the species mass fractions result directly from the solution of Eqs. (33–35) since, for this case, $z_i = Y_i/Y_{ie}$. With $f'(\eta)$, $\phi(\eta)$, and $z_i(\eta)$ known, the other flow property ratios of interest, namely, (h/h_e), (T/T_e), and (ρ_e/ρ) are determined from Eqs. (36, 39, and 40), respectively. Since the normal coordinate in the physical plane is related to η by

$$y = \frac{(2\xi)^{1/2}}{\rho_e u_e} \left[\varkappa \int_0^\eta \phi d\eta + (1 - \varkappa) \int_0^\eta f'^2 d\eta\right] \quad (42)$$

the transformation of the forementioned flow property ratios to the physical plane is completely specified once ξ is determined from Eq. (18). It is to be noted that the reference value¶ $\rho^*\mu^*$ should be selected to correspond to an average value of this product across the mixing region.

When the equilibrium flow case is considered, Eq. (35) yields the element mass fractions for each of the e elements comprising the various species of the system since for this case $z_i = z_k = \tilde{Y}_k/\tilde{Y}_{ke}$. The species mass fractions, together with the temperature for this particular case, are evaluated from the simultaneous solution of Eqs. (10, 14, and 39), which comprise a system of $(s + 1)$ equations in $(s + 1)$ unknowns. Once these flow properties are evaluated, the transformation to the physical plane proceeds along the lines outlined for the frozen flow case.

In Ref. 5, it was pointed out that the equilibrium behavior for temperatures of interest in supersonic combustion can be approximated by considering a flame-sheet combustion model. This model is based on the low-speed diffusion

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§ These assumptions imply the existence of local similarity where the external conditions are assumed to be slowly varying functions of the streamwise coordinate.

¶ An accurate description of the reference conditions would involve an analysis of the type presented in Refs. 15 and 16. In practical applications, however, it would probably suffice to use the centerline values.

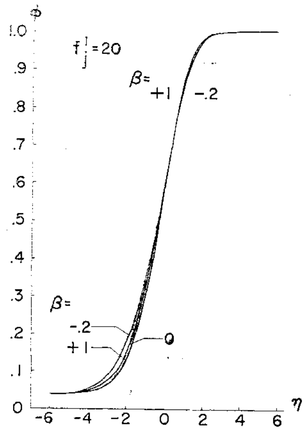

Fig. 3a Stagnation enthalpy function vs similarity parameter η (f_j' = 0.20).

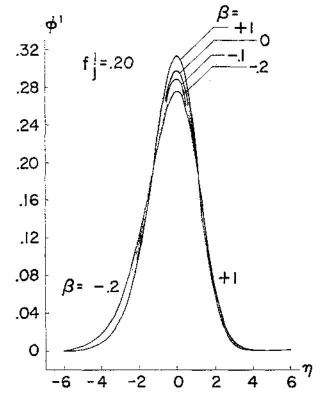

Fig. 3b Gradient of stagnation enthalpy function vs similarity parameter η (f_j' = 0.20).

flame concept applied to high-speed flows. The assumptions made are that the reaction zone is of zero thickness and that the reacting species burn instantaneously upon reaching the sheet. The first assumption reduces the reaction zone to a surface that acts as a sink for the reacting species and as a source for the combustion products. The second assumption implies that the concentrations of the reactants at the sheet must be zero. This requires that the reactants diffuse toward the sheet in the stoichiometric ratio so that no species can diffuse across the sheet to react with those on the other side. These assumptions are justified provided the reaction rates are very large compared to the diffusion velocities of the species toward the sheet.

If the hydrogen-air combustion system is considered as a typical reaction involving fuel and oxidizer, the only significant product at equilibrium conditions and low static temperatures will be water in gaseous form with nitrogen behaving as an inert dilutant.[5] For the particular combustion model assumed, $\dot{w}_i$, the mass rate term, is zero everywhere except at the flame sheet where

$$\begin{gathered} Y_1 = Y_2 = 0 \\ Y_3 + Y_4 = 1 \end{gathered} \tag{43}$$

The subscripts i = 1, 2, 3, and 4 taken here refer to molecular oxygen, hydrogen, water in gaseous form, and nitrogen, respectively. By considering the element mass fractions at the sheet, there result from Eqs. (10) and (35)

$$\tilde{Y}_{1f} = \frac{W_1}{2W_3} Y_{3f} = \frac{Y_{1e}(\phi_f - f_j'^2)}{(1 - f_j'^2)} \tag{44}$$

$$\tilde{Y}_{2f} = \frac{W_2}{W_3} Y_{3f} = \frac{(1 - \phi_f)}{(1 - f_j'^2)} \tag{45}$$

which, together with Eq. (43), yield

$$\phi_f = \frac{1 + (2W_2/W_1) Y_{1e} f_j'^2}{1 + (2W_2/W_1) Y_{1e}} \tag{46}$$

$$Y_{3f} = \frac{2W_3 Y_{1e}}{W_1[1 + (2W_2/W_1)Y_{1e}]} \tag{47}$$

$$Y_{4f} = \frac{Y_{4e}}{[1 + (2W_2/W_1)Y_{1e}]} \tag{48}$$

The velocity ratio at the flame sheet f_f' and its location in the transformed plane η_f can be determined from the basic solution $\phi = \phi(\eta)$, for the respective β, once ϕ_f is evaluated. The shape and location of the flame sheet in the physical plane follows directly by applying Eq. (42).

Method of Solution and Results

The solution of the governing nonlinear equations, i.e., Eqs. (33) and (34), subject to boundary conditions (27) and (28), is accomplished by the methods of finite-differences and linear corrections. The differential equations are approximated by central differences at discrete lattice points and result in

$$\frac{f_{n+2} - 3f_{n+1} + 3f_n - f_{n-1}}{(\Delta\eta)^3} + \left(\frac{f_n + f_{n+1}}{2}\right)\left(\frac{f_{n+2} - f_{n+1} - f_n + f_{n-1}}{2(\Delta\eta)^2}\right) = \beta\left[\left(\frac{f_{n+1} - f_n}{\Delta\eta}\right)^2 - \frac{(\phi_{n+1} + \phi_n)}{2}\right] \tag{49}$$

$$\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{(\Delta\eta)^2} + f_n \frac{(\phi_{n+1} - \phi_{n-1})}{2\Delta\eta} = 0 \tag{50}$$

The boundary conditions at ± ∞ are replaced by finite-

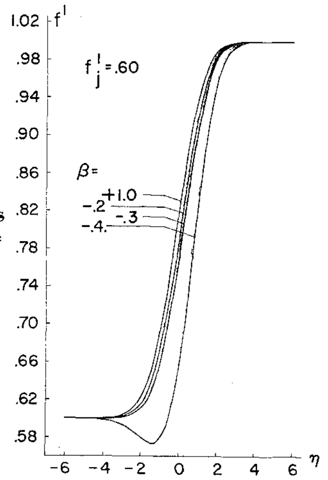

Fig. 4a Velocity profiles vs similarity parameter η (f_j' = 0.60).

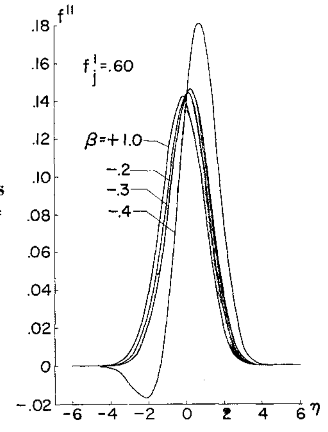

Fig. 4b Shear function vs similarity parameter η (f_j' = 0.60).

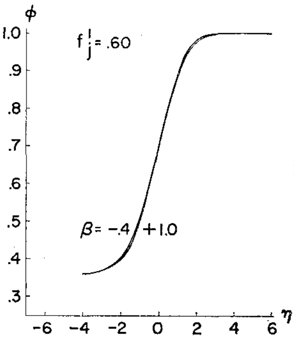

Fig. 5a Stagnation enthalpy function vs similarity parameter η ($f_j' = 0.60$).

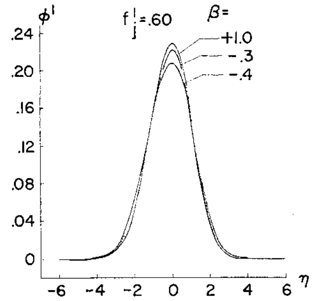

Fig. 5b Gradient of stagnation enthalpy function vs similarity parameter η ($f_j' = 0.60$).

difference conditions at the lattice points +N and −N. As a result, the boundary conditions considered become

$$\left.\begin{aligned} f_{-N+1} - f_{-N} &= f_j'\Delta\eta \\ f_0 &= 0 \\ f_N - f_{N-1} &= \Delta\eta \\ \phi_{-N} &= f_j'^2 \\ \phi_N &= 1 \end{aligned}\right\} \qquad (51)$$

The entire system consists of 4N + 2 equations; 2N − 2 equations of type (49) for $n = -N + 1$ to $N - 2$; 2N − 1 equations of type (50) for $n = -N + 1$ to $N - 1$; and the 5 boundary conditions given by (51).

This system of nonlinear equations is solved iteratively by the method of linear corrections by replacing f and ϕ in (49–51) by

$$f = \bar{f} + \tilde{f} \qquad \phi = \bar{\phi} + \tilde{\phi}$$

The quantities $\bar{f}$ and $\bar{\phi}$ are approximations to the solution, whereas $\tilde{f}$ and $\tilde{\phi}$ are first-order correction terms of which higher-order terms are neglected. The resulting linear system of equations in $\tilde{f}$ and $\tilde{\phi}$, obtained in this manner, can be solved by matrix algebra for each set of β and f_j' values selected.

Solutions of Eqs. (33) and (34), obtained by this method, have been tabulated in Ref. 9 for a range of f_j' and β values of interest. Various profiles of interest, obtained from these solutions, are presented in Figs. 2a–5b for $f_j' = 0.2$ and 0.6. It should be noted that only for $\beta < +1$ do real solutions to the differential equations exist. Larger values of the pressure gradient parameter correspond to imaginary solutions.

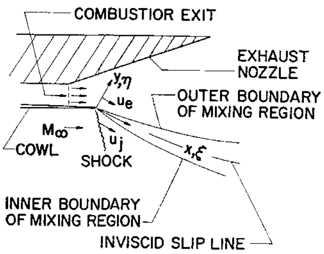

Fig. 6 Schematic representation of example case.

Table 1 Species mass fractions extenal to mixing region

	Y_{ij}	Y_{ie}
OH	0	0.0052
O_2	0.232	0.0071
H_2O	0	0.2382
H_2	0	0.0014
NO	0	0.0031
N_2	0.768	0.7445

From Figs. 2a–5b it can be seen that the pressure gradient has only a small effect on the velocity profile and almost a negligible effect on the profile of the enthalpy parameter $\phi(\eta)$. These results indicate that local similarity, as assumed in this analysis, is a valid assumption since the profiles of interest will be very close to the similar profiles corresponding to constant values of β, $\tilde{u}_e^2$, and h_{ie}/H_e. Under these conditions, it would be expected that an equation of the type given by Eq. (23) would yield fairly good results.

Indications are from Fig. 4a that the most severe changes in the profiles occur for the larger adverse pressure gradients. In particular, it can be seen from this figure that, at $\beta = -0.4$, the velocity profile varies drastically from the other profiles and exhibits a velocity reduction at the lower boundary similar to that of a separated boundary layer.

Numerical Example

In order to illustrate the applicability of the present analysis, an example calculation was performed for the case of hydrogen-air combustion products exhausting into an ambient air stream. The calculations were done for $\beta = 0$, assuming frozen flow in the mixing region. The initial conditions prior to mixing, which correspond to the exit conditions of a hydrogen-burning vehicle flying at $M_\infty = 8.5$ and 97,000 ft, were obtained from a separate analysis. Using the notation of Fig. 6, the conditions are as follows:

$$\rho_j = 2.290 \times 10^{-4} \text{ slugs/ft}^3$$

$$u_j = 5840 \text{ fps}$$

$$p_j = p_e = 0.457 \text{ atm}$$

$$T_j = 1355°\text{K}$$

$$H_j = 1400 \text{ Btu/\#}$$

$$\rho_e = 1.121 \times 10^{-4} \text{ slugs/ft}^3$$

$$u_e = 8340 \text{ fps}$$

$$T_e = 2335°\text{K}$$

$$H_e = 3360 \text{ Btu/\#}$$

whereas the mass fractions are given in Table 1.

The results of this calculation are presented in Figs. 7 and 8 where R, T/T_e, g, f', and Y_i for the various species are plotted as functions of η. For this particular case, λ =

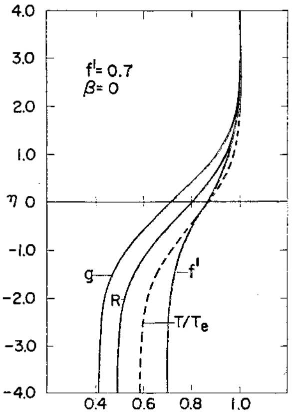

Fig. 7 Variation of flow parameters across mixing region for example case.

-0.1537 and $\kappa = 3.143$ for a value of $n = 1.11$. This value of the exponent used in Eq. (37) proved to be a very good fit to the actual variation of h_i with temperature in the temperature range of interest. It should be noted that the mass fraction of N_2 has not been included in Fig. 8 since its profile is fairly uniform varying from 0.7445 to 0.768. Since the streamwise pressure was assumed constant, ξ varies as x and η as $y/(x)^{1/2}$. Consequently, the boundaries of the mixing region will grow as $x^{1/2}$.

Conclusions

A theoretical analysis has been presented for the compressible two-dimensional mixing of two reacting gases subjected to a streamwise pressure gradient. The analysis is applicable to both limiting cases of frozen or equilibrium flows under the assumptions of unit Prandtl and Lewis numbers. An empirical law for the gas behavior within the mixing region is assumed which matches the proper density values at points across the shear layer. Within the framework of these assumptions, similar solutions have been obtained for various values of the velocity ratio f_i' and pressure gradient parameter β.

The numerical solutions to the governing equations were obtained by the utilization of an iterative procedure involving both the methods of finite-differences and linear corrections. For the sake of brevity, only some of the plots of the various profiles have been presented. A complete tabulation of these functions can be found in Ref. 9.

A numerical example of the analysis is presented for the frozen flow mixing of combustion products, issuing from the exhaust nozzle of a hypersonic hydrogen burning vehicle, with the ambient air stream. The particular case considered corresponds to a freestream Mach number of 8.5 and an altitude of 97,000 ft. The results indicate very well-behaved profiles for the flow parameters of interest and a partial check of Eq. (23), since the computed density ratio distribution checked both the equation of state and the empirical relation. A true measure of the adequacy of this relationship, however, can only be determined from experimental verification.

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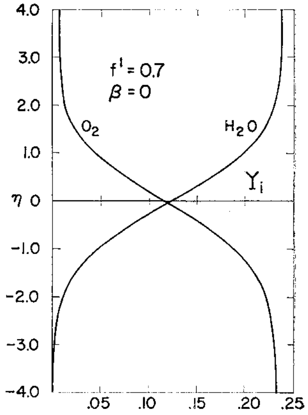

Fig. 8a Mass fraction variation of O₂ and H₂O across mixing region for example case.

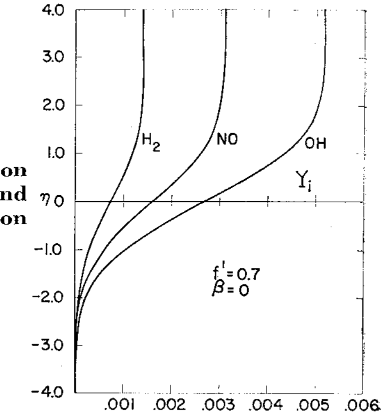

Fig. 8b Mass fraction variation of H₂, NO, and OH across mixing region for example case.

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